

MATH 2028 Honours Advanced Calculus II

2023-24 Term 1

Problem Set 4

due on Oct 13, 2023 (Friday) at 11:59PM

Instructions: You are allowed to discuss with your classmates or seek help from the TAs but you are required to write/type up your own solutions. You can either type up your assignment or scan a copy of your written assignment into ONE PDF file and submit through Blackboard on/before the due date. Please remember to write down your name and student ID. **No late homework will be accepted.**

Notations: Throughout this problem set, we use (r, θ) , (r, θ, z) and (ρ, ϕ, θ) to denote the polar, cylindrical and spherical coordinates respectively.

Problems to hand in

1. Let $\Omega \subset \mathbb{R}^2$ be the region bounded below by $y = 1$ and above by $x^2 + y^2 = 4$. Evaluate

$$\int_{\Omega} (x^2 + y^2)^{-3/2} dA.$$

2. Evaluate the iterated integral

$$\int_0^1 \int_y^1 \frac{xe^x}{x^2 + y^2} dx dy.$$

3. Find the volume of a right circular cone of base radius a and height h by integrating in cylindrical coordinates and spherical coordinates.
4. Find the volume of the region \mathbb{R}^3 inside both $x^2 + y^2 = 1$ and $x^2 + y^2 + z^2 = 2$.

Suggested Exercises

1. Find the area enclosed by the cardioid in \mathbb{R}^2 expressed in polar coordinates as $r = 1 + \cos \theta$.
2. Let $\Omega \subset \mathbb{R}^2$ be the annular region bounded by $x^2 + y^2 = 1$ and above by $x^2 + y^2 = 2$. Evaluate $\int_{\Omega} y^2 dA$.
3. Find the volume of the region in \mathbb{R}^3 bounded above by $z = 2$ and below by $z = x^2 + y^2$.
4. Find the volume of the region lying above the plane $z = a$ and inside the sphere $x^2 + y^2 + z^2 = 4a^2$ by integrating in cylindrical coordinates and spherical coordinates.
5. Find the volume of the region in \mathbb{R}^3 bounded by the cylinders $x^2 + y^2 = 1$, $y^2 + z^2 = 1$, and $x^2 + z^2 = 1$.
6. Let $\Omega \subset \mathbb{R}^3$ be the region bounded below by the sphere $x^2 + y^2 + z^2 = 2z$ and above by the sphere $x^2 + y^2 + z^2 = 1$. Evaluate the integral

$$\int_{\Omega} \frac{z}{(x^2 + y^2 + z^2)^{3/2}} dV.$$

Challenging Exercises

1. Let $\Omega \subset \mathbb{R}^n$ be a bounded subset with measure zero $\partial\Omega$. Show that for any $\epsilon > 0$, there exists a compact subset $K \subset \Omega$ such that ∂K has measure zero and $\text{Vol}(\Omega \setminus K) < \epsilon$.
2. (a) Let $\epsilon > 0$ be fixed. Show that there is a C^∞ function $g : \mathbb{R} \rightarrow [0, 1]$ such that $g(x) = 0$ for $x \leq 0$ and $g(x) = 1$ for $x \geq \epsilon$.
(b) Let $\Omega \subset \mathbb{R}^n$ be an open set and $K \subset \Omega$ be a compact subset. Prove that there exists a C^∞ function $f : \Omega \rightarrow [0, 1]$ such that $f(x) = 1$ for all $x \in K$.
3. (a) Let $S \subset \mathbb{R}^n$ be an arbitrary subset and $x_0 \in S$. We say that a function $f : S \rightarrow \mathbb{R}$ is differentiable at x_0 of class C^1 if there exists a C^1 function $g : U \rightarrow \mathbb{R}$ defined in a neighborhood U of x_0 in \mathbb{R}^n such that $g = f$ on $U \cap S$. Suppose $\varphi : \mathbb{R}^n \rightarrow \mathbb{R}$ is a C^1 function whose support lies in U . Show that the function $h : \mathbb{R}^n \rightarrow \mathbb{R}$ defined by

$$h(x) = \begin{cases} \varphi(x)g(x) & \text{when } x \in U \\ 0 & \text{when } x \notin \text{spt}(\varphi) \end{cases}$$

is a well-defined C^1 function on \mathbb{R}^n .

- (b) Prove the following statement: if $f : S \rightarrow \mathbb{R}$ is differentiable of class C^1 at each $x_0 \in S$, then f may be extended to a C^1 function $h : \Omega \rightarrow \mathbb{R}$ defined on an open subset Ω containing S .